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Determination of added fluid area in the homogenization model of beam bundles

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Abstract

A unique scalar parameter arises in the 3-D homogenization model for the beam bundle, which has the significance of the added fluid area fraction. The parameter is determined by solving a local problem defined on a unit cell, and its relation to the porosity of the bundle is investigated in this paper. This is made possible by obtaining an analytical solution of the local problem based on Weierstrass's doubly periodic functions.

Keywords: Homogenization; Interaction between solid and fluid; Beam bundle

1. Introduction

Beam bundle is composed of a large number of tubular beams, which are immersed in an acoustic fluid. The bundle can be regarded as a heterogeneous medium of periodic microstructure with a single beam and its surrounding fluid identified as a repeating element. In order to describe the dynamic behavior of the beam bundle, two approaches have been developed since the 1980s. They are the asymptotic homogenization method and the continuation approach.

Mathematical framework of the asymptotic homogenization method is based upon the work on Bensoussan et al. (1978), Sanchez-Palencia (1980), Sanchez-Palencia and Zaoui (1987) and Conca et al. (1995). Based on this, Schumann (1981a,b) and Brochard and Hammami (1991), Hammami (1990) proposed an asymptotic homogenization model for the beam bundle. Recently, a 3-D continuum model for the beam bundle was presented by Zhang (1998a,b,c). It is referred to as a "unified" model, because the two existing 2-D models of Schumann (1981a,b) and Brochard and Hammami (1991), Hammami (1990) can be considered as special cases of the 3-D model.

The first model based on the continuation approach was proposed by Shinohara and Shimogo (1981) for tubes with square or hexagonal cross-sections. In this model, the thickness of the gaps between the tubes is

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Nomenclature

$A_{\alpha\beta}$	non-added fluid area in unit cell
$B_{\alpha\beta}$	effective cross-sectional area of beam in unit cell
c	speed of sound
$D_{\alpha\beta}$	added fluid area in unit cell
F	body force of the beam
G	body force of the fluid
n_{α}	exterior unit normal to fluid domain on the interface between fluid and beam in unit cell
p	pressure of the fluid
R	radius of circular cross-section of beam in the original unit cell
r_0	radius of circular cross-section of beam in magnified unit cell
v_i	velocity of the fluid
$\|X\|$	area of domain of the original unit cell
$\|Y\|$	area of domain of magnified unit cell
w_i	displacement of the beam
ε	side length of the original square unit cell
ε_{ij}	strain of the beam
σ_{ij}	stress of the beam
ϕ	velocity potential of the fluid
κ	density ratio
λ	fluid volume fraction of the bundle
ρ	density of the fluid
$\wp(z)$	Weierstrass's elliptic function
$\zeta(z)$	Weierstrass's Zeta function
$\langle\rangle$	average value on the unit cell

Subscripts

f	fluid
s	beam structure

assumed to be small with respect to the tube's diameter. However, there is an additional term in the model, which is proportional to the second derivative of the displacement with respect to the space variables. It was found that this additional term is important in calculating the modes with significant gradients of displacements. Brochard et al. (1988) presented a homogenization technique close to the continuation method proposed by Shinohara and Shimogo. By means of this technique, the local mode, in which adjacent tubes move in opposite directions, is investigated. Cheval et al. (2001) presented an improvement and a generalization of Shinohara and Shimogo's continuation approach using a substructure technique, for tubes with square or circular cross-section, whatever the value of the gap separating two adjacent tubes. This method gives good accuracy for most kinds of tube movements.

A 3-D model was previously formulated by Zhang (1999) using pressure as the fundamental unknown in the fluid region. As a result, non-symmetry of the coefficient matrices is introduced into the corresponding finite element solution. To remove the non-symmetry, velocity potential rather than pressure is adopted as the fundamental unknown in the fluid region in the present paper. As shown by Zhang, the 3-D model has a transverse isotropic property for the circular cross-sectional beams in tandem. The homogenization equation of the model can be simplified by replacing its tensor parameters with a scalar parameter representing the added fluid area.

In this paper, an analytical solution of the local problem is presented in terms of Weierstrass's doubly periodic functions. The local solution describes the local field of the interaction of a beam and the surrounding fluid, and it gives rise to a scalar parameter in the 3-D homogenization equations. The scalar parameter has the significance of the added fluid area fraction.

2. Fundamental equations

Consider the 3-D homogenization equations given in Zhang (1999) where the fluid in the bundle is treated as compressible and non-viscous. The motion is governed by the continuity equation

$$\dot{\rho}_{\mathrm{f}} + \bar{\rho}_{\mathrm{f}} \nabla_i v_i = 0, \tag{1}$$

the momentum equilibrium equation

$$\bar{\rho}_{\mathrm{f}} \dot{v}_i = G_i - \nabla_i p, \tag{2}$$

and the equation of state

$$\dot{p} = c_{\mathrm{f}}^2 \dot{\rho}_{\mathrm{f}}. \tag{3}$$

Elastic beams in the bundle are described in terms of the equation of motion

$$\nabla_j \sigma_{ij} + F_i = \bar{\rho}_{\mathrm{s}} \ddot{w}_i, \tag{4}$$

the strain displacement relation

$$\varepsilon_{ij} = \tfrac{1}{2}(\nabla_j w_i + \nabla_i w_j), \tag{5}$$

and the continuity equation

$$\dot{\rho}_{\mathrm{s}} + \bar{\rho}_{\mathrm{s}} \nabla_i \dot{w}_i = 0. \tag{6}$$

In addition, there are interaction conditions at each interface between beams and fluid:

$$\sigma_{\alpha\beta} n_\beta = -p n_\alpha \quad \text{and} \quad v_\alpha n_\alpha = \dot{w}_\alpha n_\alpha. \tag{7}$$

Here $\nabla_i = \partial/\partial x_i$ and $x = (x_1, x_2, x_3)$ denotes a global coordinate system on a unit cell, with the x_3-axis pointing along the beam. Summation is indicated by repeated subscripts. Greek subscripts assume the value 1 and 2 while Latin subscripts range from 1 to 3. The above fundamental equations do not include the general stress–strain relation. In fact, only the tensile stress–strain relation is necessary in our approach, which is similar to the case in beam theory. Thus, we will introduce the tensile stress–strain relation together with the simple beam assumption in Section 5.

Eqs. (1)–(7), subject to appropriate boundary conditions of the whole bundle such as side wall conditions, end conditions of beams, surface and bottom conditions of fluid and initial conditions, form a closed system of equations of the dynamic problem for the beam bundle. The global behaviour of the beam bundle can be calculated by solving the proposed system. However, this system is difficult to solve due to the large number of beams. The asymptotic homogenization method provides an alternative way to establish a simple and yet rigorous mathematical model.

3. Asymptotic expansion

Consider a beam bundle in which the beams are regularly placed in the fluid. The cross-section of a single beam with its surrounding fluid is defined to be a unit cell as shown in Fig. 1. The characteristic length l of the unit cell is assumed to be much smaller than the characteristic dimension L of the bundle, so that

$$\varepsilon = \frac{l}{L} \ll 1. \tag{8}$$

A local coordinate system $y = (y_1, y_2)$ is introduced for the unit cell by

$$y_\alpha = x_\alpha/\varepsilon. \tag{9}$$

Asymptotic expansions of the unknown quantities are postulated such that

$$\begin{aligned} p &= p^{(0)}(x, y_1, y_2, t) + \varepsilon p^{(1)}(x, y_1, y_2, t) + \cdots, \\ w &= w^{(0)}(x, y_1, y_2, t) + \varepsilon w^{(1)}(x, y_1, y_2, t) + \cdots, \text{ etc.} \end{aligned} \tag{10}$$

By replacing the gradient operator ∇_α with $\nabla_\alpha + (1/\varepsilon)\partial/\partial y_\alpha$ and then equating coefficients of like powers of ε on both sides of equations, a set of expanded equations is obtained. The $O(\varepsilon^{-1})$ approximations of (1)–(7) are

$$v^{(0)}_{\alpha,\alpha} = 0, \tag{11}$$

$$p^{(0)}_{,\alpha} = 0, \tag{12}$$

$$\sigma^{(0)}_{\alpha\beta,\beta} = 0, \tag{13}$$

$$w^{(0)}_{\alpha,\beta} + w^{(0)}_{\beta,\alpha} = 0, \tag{14}$$

$$w^{(0)}_{3,\alpha} = 0, \tag{15}$$

$$w^{(0)}_{\alpha,\alpha} = 0. \tag{16}$$

The zeroth-order approximations are

$$\dot{\rho}^{(0)}_{\mathrm{f}} + \bar{\rho}_{\mathrm{f}}(\nabla_i v^{(0)}_i + v^{(1)}_{\alpha,\alpha}) = 0, \tag{17}$$

$$\bar{\rho}_{\mathrm{f}} \dot{v}^{(0)}_\alpha = G^{(0)}_\alpha - (\nabla_\alpha p^{(0)} + p^{(1)}_{,\alpha}), \tag{18}$$

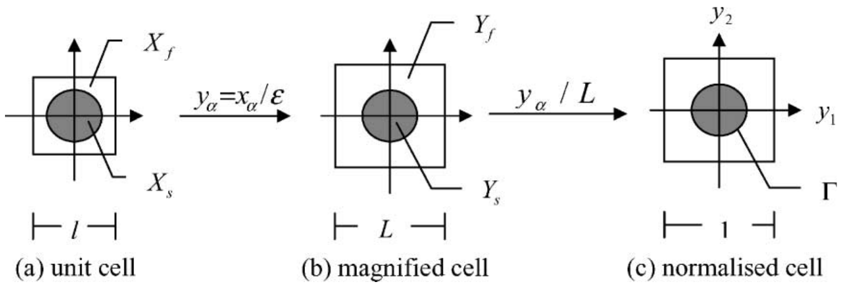

Fig. 1. Unit cells.

$$\bar{\rho}_{\mathrm{f}}\dot{v}_3^{(0)} = G_3^{(0)} - \nabla_3 p^{(0)}, \tag{19}$$

$$\dot{p}^{(0)} = c_{\mathrm{f}}^2 \dot{\rho}_{\mathrm{f}}^{(0)}, \tag{20}$$

$$\nabla_j \sigma_{\alpha j}^{(0)} + \sigma_{\alpha\beta,\beta}^{(1)} + F_\alpha^{(0)} = \bar{\rho}_{\mathrm{s}}\ddot{w}_\alpha^{(0)}, \tag{21}$$

$$\varepsilon_{33}^{(0)} = \nabla_3 w_3^{(0)}, \tag{22}$$

$$\varepsilon_{3\alpha}^{(0)} = \tfrac{1}{2}(\nabla_3 w_\alpha^{(0)} + \nabla_\alpha w_3^{(0)} + w_{3,\alpha}^{(1)}), \tag{23}$$

$$\dot{\rho}_{\mathrm{s}}^{(0)} + \bar{\rho}_{\mathrm{s}}(\nabla_i \dot{w}_i^{(0)} + \dot{w}_{\alpha,\alpha}^{(1)}) = 0, \tag{24}$$

$$\sigma_{\alpha\beta}^{(0)} n_\beta = -p^{(0)} n_\alpha, \tag{25}$$

$$v_\alpha^{(0)} n_\alpha = \dot{w}_\alpha^{(0)} n_\alpha. \tag{26}$$

The first-order approximations are

$$\varepsilon_{33}^{(1)} = \nabla_3 w_3^{(1)}, \tag{27}$$

$$\sigma_{\alpha\beta}^{(1)} n_\beta = -p^{(1)} n_\alpha, \tag{28}$$

$$v_\alpha^{(1)} n_\alpha = \dot{w}_\alpha^{(1)} n_\alpha, \text{ etc.} \tag{29}$$

Here a comma indicates differentiation with respect to local coordinates y_α.

4. Local problem

Eqs. (14)–(16) give

$$w_\alpha^{(0)} = w_\alpha^{(0)}(x,t) \quad \text{and} \quad w_3^{(0)} = w_3^{(0)}(x,t). \tag{30}$$

Moreover, from (12), (19) and (20), we have

$$p^{(0)} = p^{(0)}(x,t), \quad \rho_{\mathrm{f}}^{(0)} = \rho_{\mathrm{f}}^{(0)}(x,t), \quad v_3^{(0)} = v_3^{(0)}(x,t). \tag{31}$$

It is easily verified that the solution to the boundary value problem of (13) and (25) is

$$\sigma_{\alpha\beta}^{(0)} = -p^{(0)}\delta_{\alpha\beta}. \tag{32}$$

Differentiating (18) with respect to y_α and using (11) and (31a) yields

$$p_{,\alpha\alpha}^{(1)} = 0. \tag{33}$$

Differentiating (26) with respect to time t and then substituting (18) into the resulting equation, we obtain the interaction condition in the form

$$[G_\alpha^{(0)} - (\nabla_\alpha p^{(0)} + p_{,\alpha}^{(1)})]n_\alpha = \bar{\rho}_\alpha \ddot{w}_\alpha^{(0)} n_\alpha. \tag{34}$$

The solution to (33) that satisfies the boundary condition (34) is

$$p^{(1)} = -\chi_\alpha(y)(\nabla_\alpha p^{(0)} + \bar{\rho}_{\mathrm{f}}\ddot{w}_\alpha^{(0)} - G_\alpha^{(0)}) + \langle p^{(1)}\rangle, \tag{35}$$

where $\langle p^{(1)}\rangle$ is the average value of $p^{(1)}$ over the magnified unit cell defined by

$$\langle p^{(1)}\rangle = \frac{1}{|Y|}\int_{Y_f} p^{(1)}(x,y,t)\,\mathrm{d}y, \tag{36}$$

and the local function χ_α satisfies the following local problem in the magnified unit cell Y_f:

$$\begin{cases} \chi_{\alpha,\beta\beta} = 0, \\ \chi_{\alpha,\beta} n_\beta = n_\alpha, \\ \chi_\alpha \text{ is a doubly periodic function of } y_1 \text{ and } y_2 \text{ with period } L, \\ \langle \chi_\alpha \rangle = 0. \end{cases} \tag{37}$$

Here n_1, n_2 are the components of the exterior normal to the fluid domain on the interface between the fluid and the beam in the magnified unit cell.

In order to ascertain that the fluid in the bundle is irrotational and locally incompressible as shown by (11), the velocity field of the fluid is obtained initially from the solution of (11) and (26) as

$$v_\alpha^{(0)} = \varphi_{\alpha\beta}\langle v_\beta^{(0)}\rangle + \psi_{\alpha\beta}\dot{w}_\beta^{(0)}, \tag{38}$$

where the local functions $\varphi_{\alpha\beta}$ and $\psi_{\alpha\beta}$ satisfy respectively the following local problems in the magnified unit cell Y_f:

$$\begin{cases} \varphi_{\alpha\beta,\alpha} = 0, \\ \varphi_{\alpha\beta} n_\alpha = 0, \\ \varphi_{\alpha\beta} \text{ is the doubly periodic function of } y_1 \text{ and } y_2 \text{ with period } L, \\ \langle \varphi_{\alpha\beta} \rangle = \delta_{\alpha\beta}, \end{cases} \tag{39}$$

and

$$\begin{cases} \psi_{\alpha\beta,\alpha} = 0, \\ \psi_{\alpha\beta} n_\alpha = n_\beta, \\ \psi_{\alpha\beta} \text{ is the doubly periodic function of } y_1 \text{ and } y_2 \text{ with period } L, \\ \langle \psi_{\alpha\beta} \rangle = 0. \end{cases} \tag{40}$$

The two local functions are related by

$$\varphi_{\alpha\beta} = \delta_{\alpha\beta} - \psi_{\alpha\beta}. \tag{41}$$

The first approximation of the rotation is given by $\frac{1}{\varepsilon}\left\{v_{3,2}^{(0)}, v_{3,1}^{(0)}, (v_{2,1}^{(0)} - v_{1,2}^{(0)})\right\}$. From (31c), we have $v_{3,2}^{(0)} = v_{3,1}^{(0)} = 0$. By means of (38) and (41),

$$\begin{aligned} v_{2,1}^{(0)} - v_{1,2}^{(0)} &= (\varphi_{2\beta}\langle v_\beta^{(0)}\rangle + \psi_{2\beta}\dot{w}_\beta^{(0)})_{,1} - (\varphi_{1\beta}\langle v_\beta^{(0)}\rangle + \psi_{1\beta}\dot{w}_\beta^{(0)})_{,2} = (\varphi_{2\beta,1} - \varphi_{1\beta,2})\langle v_\beta^{(0)}\rangle + \psi_{2\beta,1} - \psi_{1\beta,2})\dot{w}_\beta^{(0)} \\ &= (\varphi_{2\beta,1} - \varphi_{1\beta,2})\langle v_\beta^{(0)}\rangle - \dot{w}_\beta^{(0)} = 0. \end{aligned}$$

$v_{3,2}^{(0)} = v_{3,1}^{(0)} = 0$ and $v_{2,1}^{(0)} - v_{1,2}^{(0)} = 0$ show that the first approximation of the rotation of the fluid field vanishes, or the fluid is locally irrotational. Hence a velocity potential $\phi^{(0)}$ may be introduced to satisfy

$$p^{(0)} = -\bar{\rho}_f\dot{\phi}^{(0)}. \tag{42}$$

5. 3-D homogenization equations

Taking the average of (17) and then eliminating $v_\alpha^{(1)}$ via (24) and (29), we can obtain one of the 3-D homogenization equations in the form

$$\left[\frac{\lambda}{c_{\mathrm{f}}^2}+\frac{1-\lambda}{\kappa c_{\mathrm{s}}^2}\right]\ddot{p}+\bar{\rho}_{\mathrm{f}}\nabla_\alpha[\langle\dot{v}_\alpha\rangle+(1-\lambda)\ddot{w}_\alpha]-\lambda\nabla_3\nabla_3 p+\lambda\nabla_3 G_3=0. \tag{43}$$

In the above derivation, (19) and (20) and the relation $\dot{p}^{(0)}=c_{\mathrm{s}}^2\dot{\rho}_{\mathrm{s}}^{(0)}$ given by Schumann (1981a,b) are taken into consideration. Here $\lambda=|X_{\mathrm{f}}|/|X|=|Y_{\mathrm{f}}|/|Y|$ is the fluid volume fraction or porosity of the bundle. $|X_{\mathrm{f}}|$ and $|Y_{\mathrm{f}}|$ are the areas of fluid in the unit cell and in the magnified unit cell respectively; $|X|$ and $|Y|$ are the areas of the unit cell and the magnified unit cell respectively. The superscript (0) has been omitted for simplicity.

$p^{(1)}$ in (18) can be eliminated by means of (35). Taking the average of the resulting equation gives the second homogenization equation

$$\bar{\rho}_{\mathrm{f}}\langle\dot{v}_\alpha\rangle=A_{\alpha\beta}(G_\beta-\nabla_\beta p)+\bar{\rho}_{\mathrm{f}}D_{\alpha\beta}\ddot{w}_\beta, \tag{44}$$

where

$$D_{\alpha\beta}=\frac{1}{|Y|}\int_{Y_{\mathrm{f}}}\chi_{\beta,\alpha}\,\mathrm{d}y, \tag{45}$$

and

$$A_{\alpha\beta}=(1-\lambda)\delta_{\alpha\beta}-D_{\alpha\beta}. \tag{46}$$

Eqs. (43) and (44) can be further simplified by eliminating the mean velocity $\langle v_\alpha\rangle$. The result is thus

$$\left(\frac{\lambda}{c_{\mathrm{f}}^2}+\frac{1-\lambda}{\kappa c_{\mathrm{s}}^2}\right)\ddot{p}-\nabla_\alpha(A_{\alpha\beta}\nabla_\beta p)-\lambda\nabla_3\nabla_3 p+\bar{\rho}_{\mathrm{f}}\nabla_\alpha(B_{\alpha\beta}\ddot{w}_\beta)=0. \tag{47}$$

The body force is considered to be constant in the above consideration so that the term $\lambda\nabla_3 G_3$ is neglected.

In order to derive the last homogenization equation, consider the identity

$$\int_{Y_{\mathrm{s}}}\sigma_{\alpha\beta,\beta}^{(1)}\,\mathrm{d}y=\int_\Gamma\sigma_{\alpha\beta}^{(1)}\,\mathrm{d}l, \tag{48}$$

where $\Gamma=Y_{\mathrm{s}}\cap Y_{\mathrm{f}}$ is the interface between the beam and the fluid domain in the magnified unit cell. Substituting (21) and (28) into (48) yields

$$\int_{Y_{\mathrm{s}}}(-\nabla_\beta\sigma_{\alpha\beta}^{(0)}-\nabla_3\sigma_{3\alpha}^{(0)}-F_\alpha^{(0)}+\bar{\rho}_{\mathrm{s}}\ddot{w}_\alpha^{(0)})\,\mathrm{d}y-\int_{Y_{\mathrm{f}}}p_{,\alpha}^{(1)}\,\mathrm{d}y=0. \tag{49}$$

Substituting (32) and (35) into (49), we obtain the result

$$(1-\lambda)(\nabla_\alpha p^{(0)}+\bar{\rho}_{\mathrm{s}}\ddot{w}_\alpha^{(0)}-F_\alpha^{(0)})+D_{\alpha\beta}(\nabla_\beta p^{(0)}+\bar{\rho}_{\mathrm{f}}\ddot{w}_\beta^{(0)}-G_\beta^{(0)})-\frac{1}{|Y|}\int_{Y_{\mathrm{s}}}\nabla_3\sigma_{3\alpha}^{(0)}\,\mathrm{d}y=0, \tag{50}$$

in which the term

$$-\frac{1}{|Y|}\int_{Y_{\mathrm{s}}}\nabla_3\sigma_{3\alpha}^{(0)}\,\mathrm{d}y=-\frac{1}{|Y|}\nabla_3\int_{Y_{\mathrm{s}}}\sigma_{3\alpha}^{(0)}\,\mathrm{d}y=-\frac{1}{|Y|}\nabla_3 V_\alpha^{(0)}, \tag{51}$$

where $V_\alpha^{(0)}$ represents the lateral resultant shear in the direction of y_α over the cross-section of the beam on the magnified unit cell.

According to beam theory, the stress–strain relation with the assumption of straight normal can be expressed in the form

$$\sigma_{33}=E\varepsilon_{33}\ \text{and}\ \varepsilon_{3\alpha}=0;\quad \alpha=1,2. \tag{52}$$

The asymptotic expansions are such that

$$\varepsilon_{3\alpha}^{(0)} = 0, \tag{53}$$

and

$$\sigma_{33}^{(0)} = E\varepsilon_{33}^{(0)} \quad \text{and} \quad \sigma_{33}^{(1)} = E\varepsilon_{33}^{(1)}. \tag{54}$$

Returning to (30) of the local problem, the zeroth-order displacements $w_\alpha^{(0)}$ and $w_3^{(0)}$ can be seen as the transverse and axial displacement of the neutral axis of beams, respectively. The axial displacement of the neutral axis is not considered, namely

$$w_3^{(0)} = 0. \tag{55}$$

Then, according to (22) and (54a), we have

$$\varepsilon_{33}^{(0)} = \sigma_{33}^{(0)} = 0. \tag{56}$$

Substituting (53) and (55) into (23), we obtain

$$w_3^{(1)} = -(\nabla_3 w_\alpha^{(0)})y_\alpha, \tag{57}$$

where $\nabla_3 w_\alpha^{(0)}$ stands for the rotation of the cross-section of the beams. This is similar to the displacement assumption in elementary theory of bending. Therefore, from (27) and (54b), the axial stress component in the beams is

$$\sigma_{33}^{(1)} = -E(\nabla_3\nabla_3 w_\alpha^{(0)})y_\alpha, \tag{58}$$

where $\nabla_3\nabla_3 w_\alpha^{(0)}$ stands for the curvature of the beams after bending, which is also similar to the result in elementary theory of bending.

According to beam theory, there is no normal stress on the cross-section of a beam, such that $\int_{Y_s} \sigma_{33}^{(1)}\,\mathrm{d}y = 0$. This requires $\int_{Y_s} y_\alpha\,\mathrm{d}y = 0$, which means that the origin of the local coordinates must be taken at the centroid of a magnified cross-section of the beam. Substituting (58) into the bending moment $M_\alpha = \int_{Y_s} \sigma_{33} y_\alpha\,\mathrm{d}y = \varepsilon \int_{Y_s} \sigma_{33}^{(1)} y_\alpha\,\mathrm{d}y$, we have

$$M_\alpha^{(0)} = -E\left(\nabla_3\nabla_3 w_{(\alpha)}^{(0)}\right) I_{(\alpha)}(y), \tag{59}$$

where the local coordinate axes y_α are the principal axes of the moment of area of the magnified cross-section of the beam. Here, $M_\alpha^{(0)}$ stands for the bending moment in $y_\alpha x_3$-plane, $I_\alpha(y)$ is the associated moment of inertia of the magnified cross-section of the beam.

As mentioned above, the local coordinate axes y_α must be the central principal axes of the magnified cross-section of the beam. This is possible by taking the symmetric axes of the magnified cross-section of the beam as the local coordinate axes y_α. Naturally, the lateral resultant shear $V_\alpha^{(0)}$ can be solved using beam theory such that

$$V_\alpha^{(0)} = \nabla_3 M_\alpha^{(0)} = -\nabla_3 E I_{(\alpha)}(y) \nabla_3\nabla_3 w_{(\alpha)}^{(0)}. \tag{60}$$

As $V_\alpha^{(0)}$ represents the lateral resultant shear over the cross-section of the magnified beam, it is natural that the average lateral resultant shear of both of the magnified beam and the real beam is identical, which is expressed in the form

$$\frac{1}{|Y|} V_\alpha^{(0)} = \frac{1}{|X|} S_\alpha^{(0)}, \tag{61}$$

where $S_\alpha^{(0)}$ stands for the lateral resultant shear of the real beam. $S_\alpha^{(0)}$ can be calculated according to (60), if we use the moment of inertia of the cross-section of the real beam $I_\alpha(x)$ instead of the magnified beam $I_\alpha(y)$.

Considering (60) and (61), the expression (50) becomes

$$M_{\alpha\beta}\ddot{w}_\beta + B_{\alpha\beta}\nabla_\beta p + \nabla_3\nabla_3\left(\frac{EI_{(\alpha)}}{|X|}\nabla_3\nabla_3 w_\alpha\right) - (1-\lambda)F_\alpha - D_{\alpha\beta}G_\beta = 0, \tag{62}$$

where

$$M_{\alpha\beta} = \bar{\rho}_{\mathrm{s}}(1-\lambda)\delta_{\alpha\beta} + \bar{\rho}_{\mathrm{f}}D_{\alpha\beta}, \tag{63}$$

$$B_{\alpha\beta} = (1-\lambda)\delta_{\alpha\beta} + D_{\alpha\beta}, \tag{64}$$

and $EI_{(\alpha)}$ is the flexural rigidity of the real beam in the $x_\alpha x_3$-plane. Again, the superscript (0) in (62) has been omitted.

Eqs. (47) and (62) are the final compact system of 3-D homogenization equations, with the term $\nabla_3\nabla_3 p$ in (43) describing the influence of the axial flow upon the deflection of the beam.

It is remarkable that only the stress–strain relation and beam simplification are used to determine the lateral resultant shear, which appears in the second homogenization equation. The components of the constant body force of the beam and the fluid, F_3 and G_3, have no influence on the equations.

6. Potential equations

Eqs. (47) and (62) are expressed in terms of the pressure, taken as the fundamental unknown in the fluid region. As a result, non-symmetry of the coefficient matrices is introduced into the corresponding finite element solution. To remove this non-symmetry, a velocity potential rather than pressure is adopted as the fundamental unknown in the fluid region.

Integrating Eq. (47) with respect to time and introducing (42) into the integration, we obtain the equation

$$\left(\frac{\lambda}{c_{\mathrm{f}}^2} + \frac{1-\lambda}{\kappa c_{\mathrm{s}}^2}\right)\ddot{\phi} - A_{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - \lambda\nabla_3\nabla_3\phi - B_{\alpha\beta}\nabla_\alpha\dot{w}_\beta = 0. \tag{65}$$

Similarly, Eq. (62) is rewritten as

$$M_{\alpha\beta}\ddot{w}_\beta - \bar{\rho}_{\mathrm{f}}B_{\alpha\beta}\nabla_\beta\dot{\phi} + \nabla_3\nabla_3\left(\frac{EI}{|X|}\nabla_3\nabla_3 w_\alpha\right) - (1-\lambda)F_\alpha - D_{\alpha\beta}G_\beta = 0. \tag{66}$$

In the axial direction (x_3-axis), the system contains fourth order derivatives of w_α and second order derivatives of ϕ. This indicates that end conditions for both ends of the beams together with one free surface condition and one bottom condition of fluid are necessary to complete the formulation of the boundary-value problem. The global behaviour of the beam bundle can then be determined by solving the boundary-value problem.

7. Scalar parameter

In the homogenization Eqs. (65) and (66), a tensor parameter $D_{\alpha\beta}$ appears. This tensor is symmetric in the two subscripts due to the fact that

$$\begin{aligned} D_{\alpha\beta} &= \frac{1}{|Y|}\int_{Y_{\mathrm{f}}}\chi_{\beta,\alpha}\,\mathrm{d}y = \frac{1}{|Y|}\int_{\Gamma}\chi_\beta n_\alpha\,\mathrm{d}l \overset{(37b)}{=} \frac{1}{|Y|}\int_{\Gamma}\chi_\beta\chi_{\alpha,\gamma}n_\gamma\,\mathrm{d}l \\ &= \frac{1}{|Y|}\int_{Y_{\mathrm{f}}}(\chi_\beta\chi_{\alpha,\gamma})_{,\gamma}\,\mathrm{d}y \overset{(37a)}{=} \frac{1}{|Y|}\int_{Y_{\mathrm{f}}}\chi_{\beta,\gamma}\chi_{\alpha,\gamma}\,\mathrm{d}y, \end{aligned} \tag{67}$$

where

$$D_{\alpha\beta} = D_{\beta\alpha}. \tag{68}$$

In addition, the local problem on the magnified unit cell has symmetry in the form

$$\chi_1(y_1, y_2) = \chi_2(-y_2, y_1) = \chi(y_1, y_2), \tag{69}$$

when both points (y_1, y_2) and $(-y_2, y_1)$ are in the fluid region Y_f. For beams with circular cross-section arranged in tandem, such symmetry is satisfied. Substituting (69) into the definition of $D_{\alpha\beta}$ (45), we have

$$D_{11} = D_{22} = D \quad \text{and} \quad D_{12} = -D_{21}. \tag{70}$$

It then follows from Eqs. (68) and (70) that the symmetric tensor $D_{\alpha\beta}$ is isotropic, i.e.

$$D_{\alpha\beta} = D\delta_{\alpha\beta}. \tag{71}$$

Subsequently, in view of (46), (63) and (64), all the following tensors are also found to be isotropic such that

$$A_{\alpha\beta} = A\delta_{\alpha\beta}, \quad B_{\alpha\beta} = B\delta_{\alpha\beta} \text{ and } M_{\alpha\beta} = M\delta_{\alpha\beta}, \tag{72}$$

where

$$A = \lambda - D, \tag{73}$$

$$B = 1 - \lambda + D, \tag{74}$$

and

$$M = \bar{\rho}_s(1 - \lambda) + \bar{\rho}_f D. \tag{75}$$

Hence, a unique scalar parameter D in the 3-D homogenization equations is obtained.

All the parameters D, A and B are non-dimensional. As can be seen from the definition of M in (75), the significance of the term $\bar{\rho}_f D$ is the added fluid mass per unit cell per unit length of the beam. Hence, D represents the added fluid area fraction. According to (74) B represents the effective area fraction of the beam, which is the total area of the beam and the added fluid divided by the area of the unit cell. Moreover, (73) indicates that A is the area fraction of the fluid that is not added onto the beam. As all the tensors $A_{\alpha\beta}$, $B_{\alpha\beta}$ and $D_{\alpha\beta}$ are isotropic, the added fluid has an equal thickness around the beam. A, B and D are illustrated in Fig. 2.

Substituting (71) and (72) into (65) and (66), we obtain the equations in velocity potential with one scalar parameter as

$$\left(\frac{\lambda}{c_f^2} + \frac{1-\lambda}{\kappa c_s^2}\right)\ddot{\phi} - A\nabla_\alpha\nabla_\alpha\phi - \lambda\nabla_3\nabla_3\phi - B\nabla_\alpha\ddot{w}_\alpha = 0, \tag{76}$$

and

$$M\ddot{w}_\alpha - \bar{\rho}_f B\nabla_\alpha\dot{\phi} + \nabla_3\nabla_3\left(\frac{EI}{|X|}\nabla_3\nabla_3 w_\alpha\right) - (1-\lambda)F_\alpha - DG_\alpha = 0. \tag{77}$$

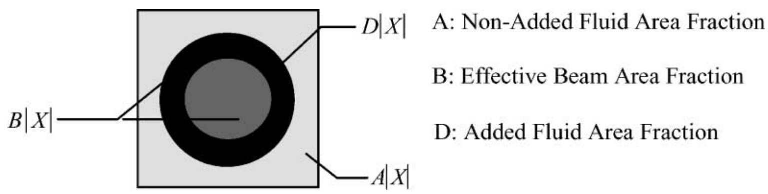

Fig. 2. Significance of A, B and D.

8. Solution of local problem

In view of (69), the local problem (37) can be rewritten as

$$\begin{cases} \chi_{,\beta\beta} = 0, \\ \chi_{,\beta} n_\beta = n_1, \\ \chi \text{ is the doubly periodic function of } y_1 \text{ and } y_2 \text{ with period } L, \\ \langle \chi \rangle = 0, \end{cases} \tag{78}$$

on the magnified unit cell of length L. The size of the cell indicates that the period of the local function $\chi(y_1, y_2)$ is L. To enable the doubly periodic Weierstrass's elliptic function with periods $\omega_1 = 1$ and $\omega_2 = i$ to express $\chi(y_1, y_2)$, the period of variables y_1 and y_2 have to be changed from L to 1. Thus, a new variable y_α/L and a new local function χ/L are introduced. However, it is not necessary to perform the transformation. Instead, one simply takes the local problem (78) and the parameter D in (45) and (71) on a normalized unit cell with side length 1 instead of a magnified unit cell with side length L. Correspondingly, Y_s and Y_f are used in the normalized unit cell.

Since Y_s is a circular domain, a polar coordinate system (r, θ) is more convenient. Therefore, the interaction condition (78b) becomes

$$\left.\frac{\partial \chi}{\partial r}\right|_{r=r_0} = \cos\theta, \tag{79}$$

where r_0 is the normalized radius of the beam.

In order to satisfy (78a), $\chi(y_1, y_2)$ is expressed as the real part of an analytic function $f(z)$, i.e.

$$\chi(y_1, y_2) = \chi(r, \theta) = \mathrm{Re}(f(z)), \tag{80}$$

in which $z = y_1 + \mathrm{i}y_2 = r e^{\mathrm{i}\theta}$. Similar to Parton and Kudryiavtsev (1993), Nie et al. (1998) proposed to express $f(z)$ in terms of the Weierstrass's functions as follows:

$$f(z) = A_0\left[z - \frac{1}{\pi}\varsigma(z)\right] + \sum_{k=0}^{\infty} A_{2k+4} r_0^{2k+4} \frac{\wp^{(2k+1)}(z)}{(2k+2)!}, \tag{81}$$

where A_0, A_{2k+4} are unknown real constants, $\wp(z)$, $\varsigma(z)$ are Weierstrass's elliptic function and Zeta function respectively. The Laurent series expansions of the Weierstrass's function can be expressed as (Chandrasekharan, 1985)

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{j=1}^{\infty} b_j z^{2j}, \\ \varsigma(z) &= \frac{1}{z} - \sum_{j=1}^{\infty} \frac{1}{2j+1} b_j z^{2j+1}, \\ \wp^{(2k+1)}(z) &= -(2k+2)!\frac{1}{z^{2k+3}} + \sum_{j=0}^{\infty} \frac{(2j+2k+2)!}{(2j+1)!} b_{j+k+1} z^{2j+1}, \end{aligned} \tag{82}$$

where

$$\begin{aligned} b_1 &= 3\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(m+\mathrm{i}n)^4}, \\ b_2 &= 5\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(m+\mathrm{i}n)^6} = 0, \end{aligned} \tag{83}$$

$$b_j = \frac{3}{(2j+3)(j-2)} \sum_{k=1}^{j-2} b_k \cdot b_{j-k-1}; \quad j = 3, 4, 5, \ldots$$

Due to the periodic properties of Weierstrass's functions, (80) satisfies (78c). It is also obvious that $f(z)$ is an odd function, the constraint condition (78d) is thus satisfied. Substitution of (81) and (82) into (79) leads to a set of algebraic equations for A_0, A_{2k+4}, $k = 0, 1, 2, \ldots$ It can be shown that the first N coefficients $F_k = A_{2k+2}$, $k = 1, N$, satisfy the following symmetrical matrix equation

$$[C](F_1, F_2, \ldots, F_N)^{\mathrm{T}} = (g_1, g_2, \ldots, g_N)^{\mathrm{T}}, \tag{84}$$

where

$$[C] = [C]^{\mathrm{T}} = \begin{bmatrix} c_{11}+3 & c_{12} & \ldots & c_{1N} \\ c_{21} & c_{22}+5 & \ldots & c_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ c_{N1} & c_{N2} & \ldots & c_{NN}+2N+1 \end{bmatrix}, \tag{85}$$

$$c_{jk} = r_0^{2j+2k+2} \left[\frac{(2j+2k)!}{(2j)!(2k)!} b_{j+k} - \frac{r_0^2}{\pi r_0^2 + 1} b_j b_k \right], \tag{86}$$

and

$$g_j = -\frac{r_0^{2j+2}}{\pi r_0^2 + 1} b_j. \tag{87}$$

Moreover, A_0 is given by

$$A_0 = \frac{\pi r_0^2}{\pi r_0^2 + 1} \left[1 - \sum_{k=1}^{\infty} F_k r_0^{2k+2} b_k \right]. \tag{88}$$

Finally, the local function χ can be expressed in terms of coefficients A_0 and F_k in the form

$$\chi(y_1, y_2) = \chi(r, \theta) = \mathrm{Re}\left[\left(A_0 + \sum_{k=1}^{\infty} r_0^{2k+2} F_k b_k \right) z - A_0 \frac{1}{\pi z} - \sum_{j=1}^{\infty} \left(\frac{A_0}{\pi} \frac{1}{2j+1} b_j + \sum_{k=1}^{\infty} \frac{(2j+2k)!}{(2j+1)!(2k)!} r_0^{2k+2} F_k b_{j+k} \right) z^{2j+1} - \sum_{k=1}^{\infty} r_0^{2k+2} F_k \frac{1}{z^{2k+1}} \right]. \tag{89}$$

Taking into account of (45) and (71), we arrive at the formula

$$D = \int_{Y_{\mathrm{f}}} \chi_{,1} \, \mathrm{d}y_1 \, \mathrm{d}y_2 = \int_{\Gamma} \chi n_1 \, \mathrm{d}s = -r_0 \int_0^{2\pi} \chi(r_0, \theta) \cos\theta \, \mathrm{d}\theta. \tag{90}$$

Substituting (88) and (89) into (90), D can now be expressed in terms of F_k and b_k as

$$D = \frac{\pi r_0^2}{\pi r_0^2 + 1} \left(1 - \pi r_0^2 - 2 \sum_{k=1}^{\infty} F_k b_k r_0^{2k+2} \right). \tag{91}$$

Finally, using the fact that

$$\lambda = \frac{|Y_{\mathrm{f}}|}{|Y|} = 1 - \pi r_0^2, \tag{92}$$

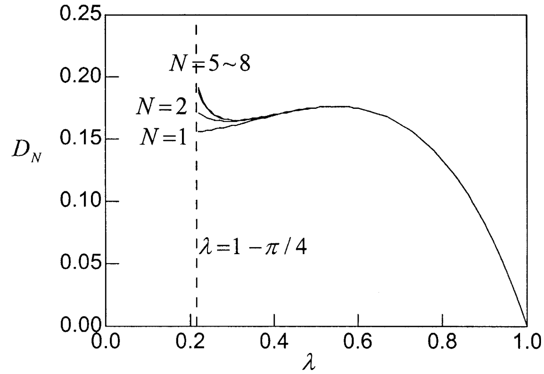

Fig. 3. Change of non-dimensional added fluid area with fluid volume fraction of the bundle.

we obtain the expression for D as a function of the porosity λ of the bundle as

$$D = \frac{1-\lambda}{2-\lambda}\left[\lambda - 2\sum_{k=1}^{\infty} F_k b_k \left(\frac{1-\lambda}{\pi}\right)^{k+1}\right]. \tag{93}$$

As $0 < r_0 \leqslant 1/2$, the range of λ is restricted to $1 - \pi/4 < \lambda \leqslant 1$.

For the purpose of comparison, numerical computations have been carried out using the following truncated form:

$$D_N = \frac{1-\lambda}{2-\lambda}\left[\lambda - 2\sum_{k=1}^{N} F_k b_k \left(\frac{1-\lambda}{\pi}\right)^{k+1}\right]. \tag{94}$$

With $N = 1, 2, 5, 6, 7, 8$, the $D_N \sim \lambda$ curves are illustrated in Fig. 3. Results show that, for the case of $N = 1$, 2, 5, 6, 7, as compared with that of $N = 8$, the maximum errors due to truncation for D are 18.16%, 10.09%, 1.91%, 0.996% and 0.398%, respectively. It indicates that $N = 5$ is sufficient for convergence of the series in (91).

In addition to the above, Zhang (1999) also derived the relationships between the equivalent sound speed and porosity. In the case where the beams are fixed, the equivalent sound speed is given by

$$c_{\mathrm{eq}} = c_{\mathrm{f}}\sqrt{1 - \frac{D}{\lambda}}, \tag{95}$$

and in the case where the beams move in phase with the fluid, the relationship is then

$$c_{\mathrm{eq}} = c_{\mathrm{f}}\sqrt{\frac{2}{1-\lambda}}, \quad \text{for } 1 - \lambda \ll 1. \tag{96}$$

9. Conclusion

A 3-D homogenization model for the beam bundle is given in this paper. The model is formulated in terms of velocity potential. Only the stress–strain relationship and beam simplifications are needed to determine the lateral resultant shear, which appears in the second homogenization equation. The axial components of the constant body force of the beam and the fluid have no influence on the equations. For the circular cross-sectional beams arranged in tandem, symmetry gives rise to a scalar parameter in the governing equations having the significance of the added fluid area fraction per single beam. In order to determine this parameter, a local problem is solved and a closed-form series solution is obtained. The proposed method is based on Weierstrass's periodic functions, which can be used to express the solution as a rapidly convergent series. In addition, it is found that the scalar parameter D is directly related to the porosity λ of the bundle.

Based on the development in this paper, it will be interesting to extend to bundles with tubes in other configurations. Also, it would be appropriate to point out that limitation exists in homogenization methods and it would be necessary to carry out comparisons with experimental data or heterogeneous calculations. These will form part of the authors' future work.

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References

Bensoussan, A., Lions, J.-L., Papanicolaou, G., 1978. Asymptotic Analysis for Periodic Structures. North-Holland, Amsterdam.

Brochard, D., Hammami, L., 1991. An homogenization method applied to the seismic analysis of LMFBR cores, SMiRT 11 Transaction, vol. E, Tokyo, Japan, pp. 497–503.

Brochard, D., Jedrzejewski, F., Gibert, R.J., Chevaugeon, N., 1988. Fluid structure interaction in tube bundles: analysis of tube movements in opposite directions, Proceedings of ASME PVP Conference, 366, pp. 179–186.

Chandrasekharan, K., 1985. Elliptic Functions. Springer Verlag, Berlin.

Cheval, K., Gibert, R.J., Broc, D., Brochard, D., 2001. Fluid structure interaction in tube bundles using homogenization technique, Proceedings of ASME PVP Conference, 421, pp. 95–102.

Conca, C., Planchard, J., Vanninathan, M., 1995. Fluids and Periodic Structures. Wiley, New York.

Hammami, L., 1990. Etude de l'interaction fluide structure dans les faisceaux de tubes par une methode d'homogenisation, These, Universite Paris VI.

Nie, G.H., Hou, S.H., Chan, C.K., 1998. Seismic analysis of nuclear reactor cores: solution of a local problem, Technical Report, Department of Applied Mathematics, The Hong Kong Polytechnic University.

Parton, V.Z., Kudryiavtsev, B.A., 1993. Engineering Mechanics of Composite Structures. CRC Press Inc., USA.

Sanchez-Palencia, E., 1980. Non-homogeneous Media and Vibration Theory. Springer Verlag, Berlin.

Sanchez-Palencia, E., Zaoui, A., 1987. Homogenization Techniques for Composite Media, Lecture Notes in Physics 272. Springer Verlag, Berlin.

Schumann, U., 1981a. Homogenized equations of motion for rod bundles in the fluid with periodic structure. Ingenieur-Archiv 50, 203–216.

Schumann, U., 1981b. Virtual density and speed of sound in a fluid-solid mixture with periodic structure. International Journal of Multiphase Flow 7, 619–633.

Shinohara, Y., Shimogo, T., 1981. Vibrations of square and hexagonal cylinders in a liquid. Journal of Pressure Vessel Technology 103, 233–239.

Zhang, R.J., 1998a. A unified 3-D homogenization model of beam bundle in fluid. ASME Journal of Pressure Vessel Technology 120, 56–61.

Zhang, R.J., 1998b. Structural homogenized analysis for nuclear reactor core. Nuclear Engineering and Design 183, 151–156.

Zhang, R.J., 1998c. Homogenization model for fluid structure interaction in nuclear reactor core, ASME/JSME Joint Pressure Vessel and Piping Conference, USA.

Zhang, R.J., 1999. A beam bundle in a compressible inviscid fluid. Journal of Applied Mechanics 66, 546–548.